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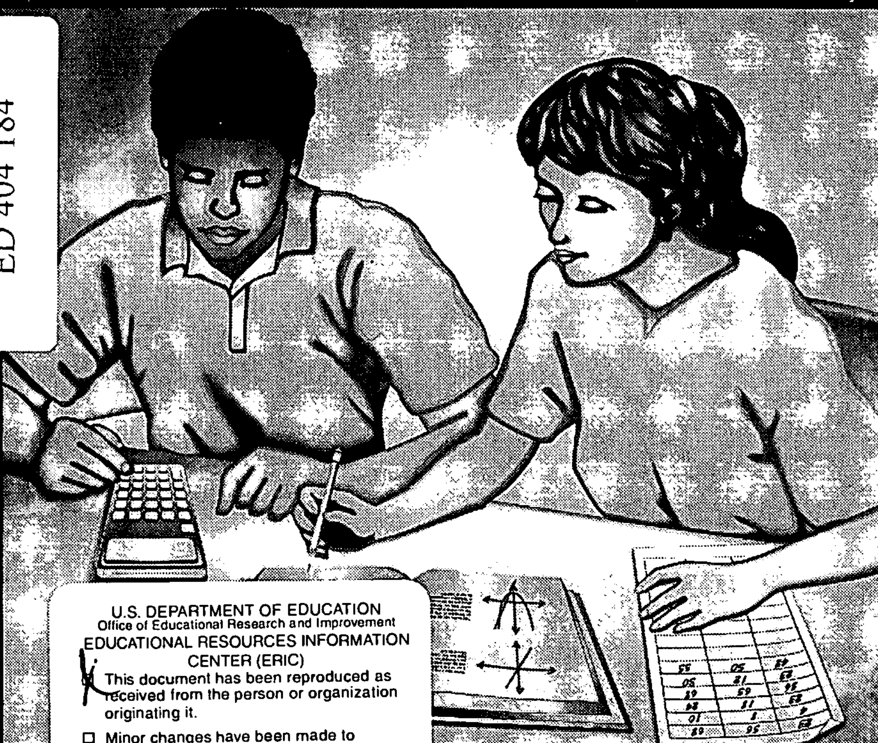
ABSTRACT

The National Assessment of Educational Progress (NAEP) is a congressionally mandated project of the U.S. Department of Education's National Center for Education Statistics. It assesses what students in the United States should know and be able to do in geography, reading, writing, mathematics, science, U.S. history, the arts, civics, and other academic subjects. This document presents a framework and recommendations for the 1996 NAEP mathematics assessment that are intended to reflect recent curricular emphases and objectives; include what various scholars, practitioners, and interested citizens believe should be in the assessment; and maintain ties to prior assessments to permit the reporting of trends in student achievement across time. Chapters include: (1) "Overview of Recommendations"; (2) "Framework for the Assessment" which discusses content strands, mathematical dimensions, families of items, percentage of items, item balance, calculators, and manipulatives; (3) "1996 NAEP Mathematics Objectives" which includes mathematical content areas and assessment strands; number sense, properties, and operations; measurement; geometry and spatial sense; data analysis, statistics and probability; and algebra and functions; (4) "Cognitive Abilities" which highlights mathematical power and mathematical abilities including conceptual understanding, procedural knowledge, and problem solving; and (5) "Item Types" which discusses multiple-choice items, open-ended items, extended open-ended items, and scoring extended open-ended items. Contains 12 references. (JRH)

Mathematics Framework for the 1996 National Assessment of Educational Progress



NAEP Mathematics Consensus Project

National Assessment Governing Board
epartment of Education

What Is NAEP?

The National Assessment of Educational Progress (NAEP) is a congressionally mandated project of the U.S. Department of Education's National Center for Education Statistics. It assesses what U.S. students should know and be able to do in geography, reading, writing, mathematics, science, U.S. history, the arts, civics, and other academic subjects. Since 1969, NAEP has surveyed the achievement of students at ages 9, 13, and 17 and, since the 1980s, in grades 4, 8, and 12.

Measuring educational achievement trends over time is critical to measuring progress toward the National Education Goals.

The National Assessment Governing Board

The National Assessment Governing Board (NAGB) was created by Congress to formulate policy for the National Assessment of Educational Progress (NAEP). Among the Board's responsibilities are developing objectives and test specifications, and designing the assessment methodology for NAEP.

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Mathematics Framework for the 1996 National Assessment of Educational Progress

NAEP Mathematics Consensus Project

Developed under contract number RN91084001 by The College Board for the National Assessment Governing Board.

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Chapter One

Overview of Recommendations

Introduction

Since 1973, the National Assessment of Educational Progress (NAEP) has gathered information about levels of student proficiency in mathematics and the related practices of teachers in our nation's schools. These periodic assessments, *The Nation's Report Card*, are published in an attempt to inform citizens about the nature of students' comprehension of the subject, to inform curriculum specialists about the level and nature of student understanding, and to inform policy makers about factors related to schooling and its relationship to student proficiency in mathematics.

Based on these surveys of students at the end of the primary, junior high, and high school levels, *The Nation's Report Card* has provided comprehensive information about what students in the United States know and can do in the area of mathematics, as well as in a number of other subject areas. These reports present information on strengths and weaknesses in students' understanding and their ability to apply that understanding in problem-solving situations; provide comparative student data according to race/ethnicity, type of community, and geographic region; describe trends in student performance across the years; and report relationships between student proficiency and certain background variables.

Context for Planning the 1996 Assessment

The National Assessment Governing Board (NAGB), created by Congress in 1988, is responsible for formulating policy for NAEP. NAGB is specifically charged with developing assessment objectives and test specifications through a national consensus approach, identifying appropriate achievement goals for each age and grade, and carrying out other NAEP policy responsibilities. In 1990, the U.S. Department of Education conducted the first voluntary state-

by-state assessment of mathematics as an adjunct to its periodic NAEP national assessments of mathematics. The 1990 state-level trials were limited to the 8th grade. In 1992, the second voluntary state-level assessments associated with NAEP were carried out at the 4th and 8th grades in mathematics, and in reading at grade 4.

To prepare for the 1990 trial state assessment, a contract was awarded in 1987 to the Council of Chief State School Officers (CCSSO) to design a framework for the assessment. The CCSSO project gave special attention to the nature of formal state objectives and frameworks for mathematics instruction. In doing so, that panel sampled state-, district-, and school-level objectives; examined the curricular frameworks on which previous NAEP assessments were based; consulted with leaders in mathematics education; and reviewed a draft version of The National Council of Teachers of Mathematics (NCTM) *Curriculum and Evaluation Standards for School Mathematics*. This project resulted in the "content-by-mathematical-ability" matrix design used to guide both the 1990 and 1992 NAEP mathematics assessments. The design was reported in *Mathematics Objectives: 1990 Assessment*.

To prepare for the next NAEP mathematics assessment, NAGB awarded a contract in fall 1991 to the College Board to develop assessment and item specifications for the upcoming mathematics assessment, which would better align the NAEP program in mathematics with the National Council of Teachers of Mathematics *Standards* (NCTM, 1989) and the *Professional Standards for Teaching Mathematics* (NCTM, 1991).

The process of developing the recommendations for the planned 1994 NAEP mathematics assessment occurred between September 1991 and March 1992. Due to a budget shortfall, however, both the new NAEP mathematics and science assessments were rescheduled from 1994 to 1996.

The NAEP mathematics project conducted by the College Board had two primary purposes. The first was to recommend a framework for the overall design of the mathematics assessment; that is, a structure for describing what students should know and be able to do in mathematics. The second was to develop specifications for the assessment items, with particular attention to a mix of formats, the item distribution for content areas within mathematics, and the conditions under which items are presented to students (e.g., use of manipulatives, use of calculators, and time to complete).

The new NAEP Mathematics Framework was considered in light of the three NAEP achievement levels—basic, proficient, and advanced. These levels are intended to provide descriptions of what students *should* know and be able to do in mathematics. Established for the 1992 mathematics scale through a broadly inclusive process and adopted by NAGB, the three levels per grade are a major means of reporting NAEP data. The new mathematics assessment was constructed with these levels in mind to ensure congruence between the levels and the test content.

The Consensus Approach

The College Board convened a Steering Committee representing national education organizations and agencies to review the direction and scope of the project. A Planning Committee of mathematics educators met to draft the Assessment Framework. Both committees considered the status of national reform efforts in mathematics education and assessment evaluations of the NAEP trial state assessment in mathematics (Silver, Kenney, and Salmon-Cox, 1991); and the fit between NAEP assessments and the contemporary teaching of mathematics at grades 4, 8, and 12 in our nation's schools (Romberg, Wilson, Smith, and Smith, 1991). Committee members are listed in the appendix.

The Planning Committee began by reviewing the framework used in the 1990 and 1992 NAEP assessments. They also made use of the findings of evaluation studies concerning the NAEP assessments. The findings of these studies were merged with recent suggestions for the assessment of student proficiency in mathematics from the mathematics education and assessment communities. The Planning Committee also analyzed the 1990 and 1992 NAEP Mathematics Framework in light of the recommendations in the NCTM *Standards* and modifications in state curriculum outlines and assessment programs precipitated by that document. Finally, the Committee reviewed information provided by the 1990 assessment, noting features of the Framework and how they assisted or hindered the clear understanding of what students knew and were able to do in mathematics appropriate to their ages and levels of education. Another important phase in the consensus process involved conducting a national mail review and convening focus groups in six states to gather input on the Committee's recommendations.

The suggested revisions in the Framework for the new NAEP assessment in mathematics are intended to reflect recent curricular emphases and objectives; to include what various scholars, practitioners, and interested citizens believe should be in the assessment; and to maintain ties to prior assessments to permit the reporting of trends in student achievement across time.

Recommendations for the 1996 NAEP Mathematics Assessment

As a result of analysis and review, the Steering Committee and Planning Committee endorsed the following recommendations for the 1996 NAEP Mathematics Assessment:

1. Content Strands

The matrix framework employed in the 1990 and 1992 NAEP assessments should be discontinued in favor of a model consisting primarily of the five major content strands used in that matrix model. Evaluation studies of the NAEP trial state assessment and other cognitive science recommendations dealing with assessment suggest that forcing content into a rigidly structured content-by-ability-level matrix distorts the nature of the discipline. A model that calls for the assessment of knowledge in discrete content-by-ability-level categories is inappropriate in an era where other, more progressive recommendations call for attention to a student's ability to connect knowledge in one area of mathematics with knowledge and abilities in other areas of mathematics.

Therefore, the recommendation was to use the five major strands: (1) Number Sense, Properties, and Operations; (2) Measurement; (3) Geometry and Spatial Sense; (4) Data Analysis, Statistics, and Probability; and (5) Algebra and Functions. In establishing these five strands, the names of the first and third strands were modified to better reflect the nature of the content relative to recommendations made in the NCTM *Standards*. The nature of the strands is further discussed in Chapters Two and Three.

2. Mathematical Abilities

The levels of mathematical ability (conceptual understanding, procedural knowledge, and problem solving) should not be

used to define specific percentages of items in each of the five content strands as had been done in the 1990 and 1992 assessments. However, these descriptors, along with the more encompassing process goals of reasoning, connections, and communication should play a central role in defining item descriptors and in achieving a balance across the task sets for each of the grade levels in the 1996 NAEP assessment in mathematics. This recommendation is discussed further in Chapters Two and Four.

3. Percentage of Items

The percentage of items allotted to each of the five strands should continue the move, begun with the 1990 assessment, toward a balance among the five strand areas and away from an assessment dominated by number and operations. The recommendations, while still retaining a core of items that reflect traditional goals in the basic skills, provide a continued movement toward a broad algebra- and geometry-oriented program at the 8th- and 12th-grade levels. The specific percentage of items recommended is further discussed in Chapter Two.

4. Item Families

To measure the breadth and depth of student knowledge in mathematics, "families" of tasks/items should be created for each grade/age level of the assessment. A "family" of tasks/items is a related set of assessment tasks that can probe the vertical or horizontal nature of a student's understanding. A vertical family might include items that measure students' abilities to give a concept definition, apply the concept in a familiar setting, use the concept or related principles to solve a new problem, and ultimately, generalize knowledge about the concept or related principles to represent a new level of understanding. A vertical family might lie within a single grade level or extend across grade levels. Another family of items might measure students' horizontal understanding of a concept or principle across the various content strands. For example, students' proficiency in solving the proportion $2/3 = 16/x$ might be measured in a number context, in a measurement setting, in a geometry setting, in a probability setting, and in an algebraic setting. Students' ability to work

with the proportion in each of these contexts tells a great deal about the richness of their understanding of the concept and the related procedural skills.

5. Constructed-Response Items

The number of items requiring student construction of the response should be increased as much as possible within the bounds of the statistical design used to carry out the assessment. The evaluation studies associated with the 1990 assessment found that the items requiring student construction of the response were most in keeping with the spirit of the NCTM *Standards*. Further, these items provide excellent opportunities to see students' abilities to reason, connect, and communicate their knowledge of mathematics. In particular, the number of extended open-ended items should be increased from the number given in the 1992 assessment.

6. Special Studies

At the 12th grade, a special study should be carried out using graphing calculators to establish baseline data for the gradual entry into the curriculum of these tools that assist students in visualizing algebraic relations.

In keeping with NCTM recommendations on hand calculator usage, NAGB should consider the unrestricted use of calculators on all phases of the assessment that are not designed to measure trends or to measure students' basic fact, operation, and estimation abilities.

7. Manipulatives

The assessment should continue to utilize reasonable manipulative materials, where possible, in measuring students' ability to represent their understandings and to reason in problem-solving situations. Such manipulative materials and accompanying tasks should be carefully chosen to cause minimal disruption of the test administration process.

8. Item Bias Review

While bias analysis is consistently done on NAEP items and performance as mandated by law, these recommendations for

shifting the types of items used on the assessment require an especially careful look at potential item bias. Data should be gathered, during field testing and during the actual assessment, regarding possible types of unforeseen item bias that may arise from incorporating less widely used types of assessment items. The 1996 and future NAEP assessments will incorporate awareness of this critical consideration, especially related to students' previous opportunities to learn and students' experience and background, both in and outside of school. Sensitivity and a sound research base will guide not only test construction but also the reporting of student performance.

These recommendations are made in an attempt to move the NAEP Mathematics Framework closer to the context of the NCTM *Curriculum and Evaluation Standards for School Mathematics*, while maintaining an adequate base for the study of trends in student achievement across time. They reflect the increasing realization that student proficiency in mathematics is not the result of the interaction of discrete cells of knowledge with a discrete list of special mathematical abilities. Rather, student proficiency in mathematics results from broad experience in forming networks of connections among mathematical ideas and skills. The Framework and specifications reflect a much more holistic and integrated view of school mathematics than the previous NAEP frameworks. As such, the Framework moves NAEP closer to the ideal described in the NCTM *Standards*.

Chapter Two

Framework for the Assessment

Content Strands

This chapter further discusses the rationale for recommendations presented in Chapter One. As in the Framework for the 1990 and 1992 mathematics assessments, the Framework for the 1996 mathematics assessment is anchored in broad strands of mathematical content reflecting the content standards in the NCTM's *Curriculum and Evaluation Standards for School Mathematics*. These content strands are:

- Number Sense, Properties, and Operations.
- Measurement.
- Geometry and Spatial Sense.
- Data Analysis, Statistics, and Probability.
- Algebra and Functions.

These strand divisions are not intended to separate mathematics into discrete elements. Rather, they are intended to provide a helpful classification scheme that describes the full spectrum of mathematical content. At the same time, content descriptions alone cannot fully express the kind of mathematical thinking described in the *Standards* and suggested by current mathematical education reforms. Even though the *Standards* that call for the development of the processes of problem solving, communicating, reasoning, and connecting knowledge of mathematics are subsumed in each of these content strands, an additional assessment dimension is needed.

Mathematical Dimensions

Previous NAEP mathematics assessments made use of matrix frameworks to specify items by both content strand and mathematical ability, as shown in figure 1. The use of such frameworks

Figure 1. Framework for the 1990 and 1992 Mathematics Assessments

Mathematical Abilities	Content Areas				
	Numbers & Operations	Measurement	Geometry	Data Analysis, Statistics, & Probability	Algebra & Functions
Conceptual Understanding					
Procedural Knowledge					
Problem Solving					

provided strong guidance for the construction of the tests in terms of breadth. Nonetheless, this type of structure has tended to work against the curricular goal of integrating mathematical knowledge across topics.

Additionally, on secondary analyses of the NAEP items, expert panels often had difficulty replicating the assignment of items to cells of the matrix on the basis of the mathematical ability classifications. Classifications varied with the rater's conceptions of children at grades 4, 8, or 12 rather than with the definitions of the mathematical abilities. The strict application of the mathematical abilities classifications in conjunction with the content strands led to a force fit of items to achieve balance across the two-dimensional matrix rather than to match the goals of mathematics education.

In real life, few mathematical situations fall clearly in one content strand or another, and few naturally reflect only one facet of mathematical thinking. Yet, to ensure a broad scope in test construction, items must be classified in a number of ways. To address this issue of item classification, the Framework for the 1996 Mathematics Assessment focuses primarily on the mathematical content strands, with additional specifications related to an assessment dimension referred to as "mathematical power," as shown in figure 2.

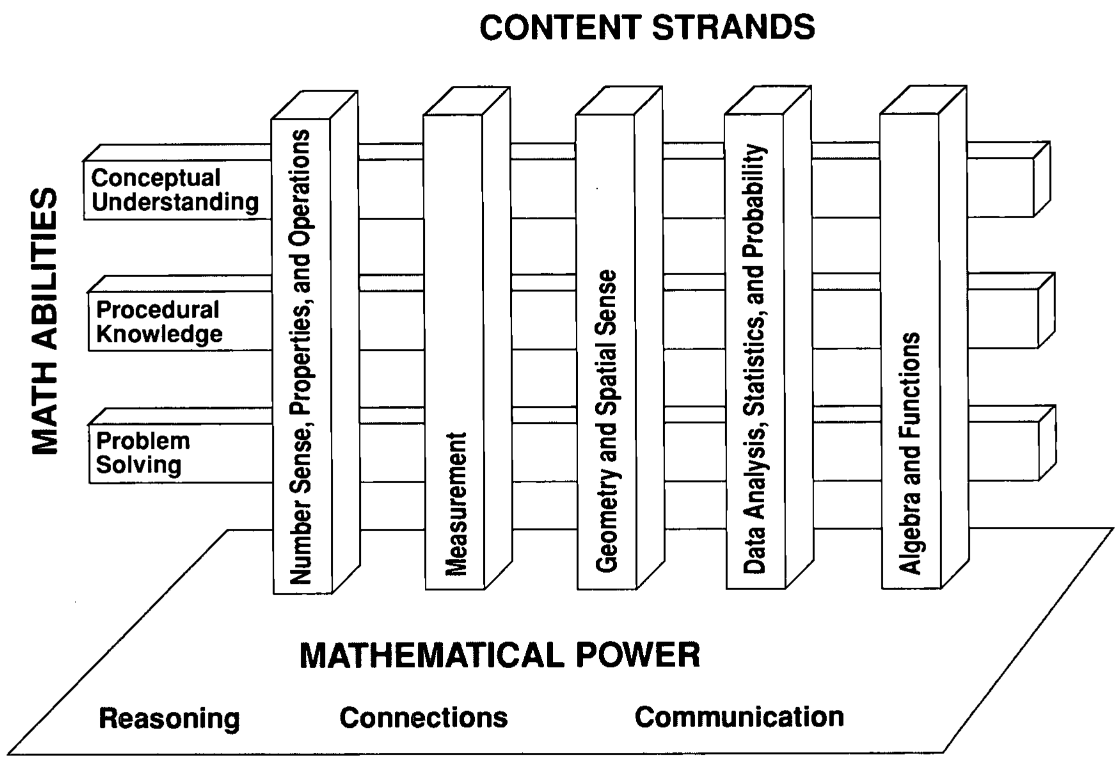


Figure 2. Mathematical Framework for the 1996 Assessment

Figure 2 shows that the curriculum is conceived as consisting of content drawn from five broad mathematical areas. Items are classified according to the major area(s) they address including both mathematical abilities and mathematical power. Mathematical power is conceived as consisting of mathematical abilities (conceptual understanding, procedural knowledge, and problem solving) within a broader context of reasoning and with connections across the broad scope of mathematical content and thinking. Communication is viewed as both a unifying thread and a way for students to provide meaningful responses to tasks.

Over the past two NAEP administrations, the conception of mathematical power as reasoning, connections, and communication has played an increasingly important role in measuring student achievement. In 1990, the assessment included short-answer, open-ended items as a way to begin to address mathematical communication. The extended open-ended items included on the 1992 assessment required students not only to communicate their ideas but also to begin to demonstrate the reasoning they used to solve problems. The 1996 assessment items will focus even more attention on mathematical power by continuing deliberate attention to reasoning and communication and by providing students with opportunities to connect their learning across mathematical content strands. These connections will be addressed within individual items that are designed to tap more than one content strand or more than one ability, as well as across items through the use of item families.

Families of Items

Families of related items will be designed to sample the depth of students' knowledge within a particular strand and students' ability to deal with concepts, principles, or procedures across content strands. Within a family, items may cross content areas, mathematical abilities, and/or grade levels. This type of grouping in the design of the assessment provides for a more indepth analysis of student performance than would a collection of discrete items. Individual student performance, comparisons of student performance across grade levels and strands, and comparisons of student performance across assessments with respect to a family of items will provide another way of looking at areas of strength and weakness.

A more detailed discussion of the nature of content in each of the strands is provided in Chapter Three, and more detailed descriptions of item types and families of items are provided in Chapter Five.

Percentage of Items

The distribution of items among the various mathematical content strands is a critical feature of the assessment design, as it reflects the relative importance and value given to each of the curricular content strands within mathematics. Over the past six NAEP assessments in mathematics, the categories have received differential emphasis, and the differentiation is continued in the Framework for the 1996 assessment. The recommended distribution of items to the strands continues to move toward a more even balance among the strands and away from the earlier model where items reflecting number facts and operations controlled more than 50 percent of the assessment item bank.

Another significant difference in the 1996 assessment is that items may be classified in more than one strand. In addition to describing minimum percentages of the item pool that should address each strand, note that maximum percentages are listed for the *Number Sense, Properties, and Operations* strand to ensure that the balance is maintained. Table 1 provides the recommended mix of items in the assessment by content strand for each of grades 4, 8, and 12 in the 1996 assessment.

These guidelines for balance present a minimum target for representation across mathematical content strands. For *Number Sense, Properties, and Operations*, notice that a maximum target is also provided. This is intended to communicate the concern that the assessment continue to shift away from a narrow number and computation focus to a more comprehensive view of mathematics. An item should be classified according to its predominant strand; it may be classified under two or more strands if it addresses substantive content from more than one area. In fact, at least half of the new items in the 1996 assessment should have major elements drawn from more than one strand, and they should be categorized in those strands. This means that the percentages listed in table 1, when translated into data on the actual item pool, will result in a percentage of items greater than those listed and will add up to more than 100 percent. Because the 1992 scale is to be maintained

Table 1. Minimum Percentage Distribution of Items by Grade and Content Strand

(An item may be classified in more than one category)

Content Strand	Grade 4	Grade 8	Grade 12
Number Sense, Properties, and Operations* (mininum/maximum)	40/70	25/60	20/50
Measurement	20	15	15
Geometry and Spatial Sense	15	20	20**
Data Analysis, Statistics, and Probability	10	15	20
Algebra and Functions	15	25	25

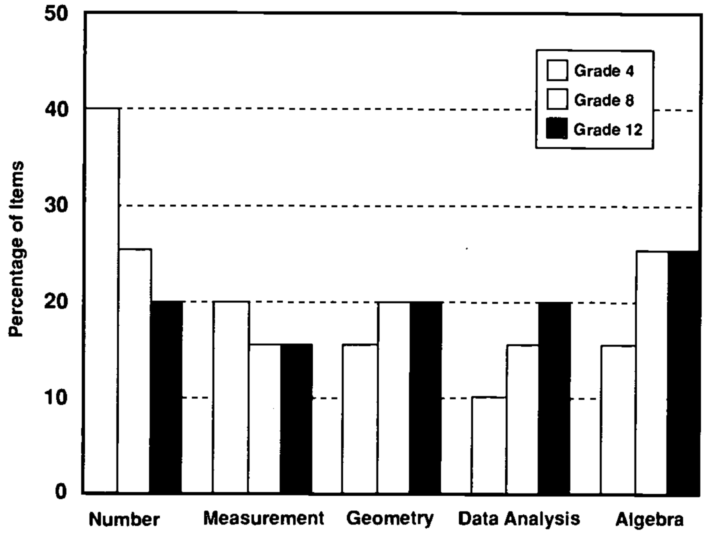
*At least half of the items in Number Sense, Properties, and Operations at each grade level should involve some aspect of estimation or mental mathematics. No more than the specified maximum percent of the items at any grade level should have a major classification in this strand.

**At grade 12, 25 percent of the items in the Geometry strand should involve topics in coordinate geometry.

in the 1996 assessment, these percentages may need to be altered slightly if field test data indicate any significant change in the scale from 1992 to 1996. If such an alteration is necessary, it is critical to ensure, to the extent possible, that the 1996 assessment reflects the levels of emphases described above. Additionally, the number of items reflecting connections among strands should continue to increase in subsequent assessments to move NAEP assessments ever closer to the vision embodied in the NCTM curriculum *Standards*—that of a student having the opportunity to demonstrate mathematical power in a broad variety of rich situations requiring connections within mathematics and with other disciplines.

The graph in figure 3 shows the relative balance given the five strands as students in the three grade levels encounter the 1996 NAEP Mathematics Assessment. The emphasis given to Number Sense, Properties, and Operations in grade 4 shifts toward growing emphases in Geometry and Spatial Sense, Data Analysis, and Algebra and Functions in the later grades.

Figure 3. Balance Between Content Strands for Percentage of Items (Minimum Percentages Shown)



Item Balance

Mathematical power can be thought of as an extension of 'mathematical abilities,' as the term was used in the 1990 and 1992 mathematics assessments. The mathematical abilities described in the Framework for these previous assessments (procedural knowledge, conceptual understanding, and problem solving) specifically address aspects of knowing and doing mathematics. Nonetheless, the development of assessment items based only on a rigid content-by-process matrix has led to contrived separation and artificial contexts. Indeed, expert reviewers of the 1990 assessment often were unable to agree on the best placement for some items in the framework matrix.

The 1996 specifications are designed to incorporate the overarching NCTM *Standards* for problem solving, communicating, reasoning, and connecting, as well as the NCTM assessment categories of conceptual understanding, procedural knowledge, and problem solving. The following recommendations are intended to guide the development of actual items for the 1996 NAEP

Mathematics Assessment. These guidelines are provided to assist in reviewing the overall balance in the assessment and to ensure that the assessment reflects some balance among "knowing that or knowing about," "knowing how," and "solving problems," within an overall demonstration of mathematical thinking in a variety of situations. Chapter Four includes a more indepth discussion of mathematical power, mathematical abilities, and additional aspects of mathematical thinking, as these relate specifically to the new mathematics assessment.

Guidelines for the balance among the conceptual understanding, procedural knowledge, and problem solving classifications should be evaluated only against the total item package at each grade level, not across each individual strand. As in the content classification, classification according to these three mathematical abilities need not, in fact should not, be forced into individual categories. Rather, an item will likely include elements of more than one of these three, and it should be classified in as many of these categories as is appropriate for the major thought processes required.

At each grade level, at least one-third of the items should be classified as conceptual understanding, at least one-third should be classified as procedural knowledge, and at least one-third should be classified as problem solving. Items with a major element of procedural knowledge in addition to either conceptual understanding or problem solving should not comprise the majority of items at any grade level.

To present a more complete picture of national mathematics performance, there should be an increase in the total number of items in the assessment and the number of items requiring student-constructed responses. In particular, any increase should reflect at least a doubling of the number of extended open-ended items contained in the 1992 NAEP assessment and an attempt to equalize the number of questions requiring students to produce a short answer with the number of multiple-choice items. This increased number of items will also allow the extension of grade 12 content into the precalculus level, not previously assessed.

The percentage distributions presented here, the lists of topics provided in Chapter Three, and the described elements of mathematical power are not intended to prescribe curriculum standards; rather, they are designed for the purpose of constructing a complete

and balanced assessment instrument reflecting the NCTM *Standards* and practice in mathematics education at each grade level. An analysis of student performance across all of the items will permit NAEP to report on average mathematics proficiency. In addition, analysis of performance on subsets of items will permit reporting on patterns of achievement in each of the five strands, as well as in procedural knowledge, conceptual understanding, and problem solving.

Calculators

In past NAEP assessments, students have been provided calculators to gather information on special blocks of items measuring ability to use calculators in mathematical situations. With the 1996 assessment, NAEP should investigate unrestricted use of calculators on all but trend blocks or specifically excluded items at each grade level. Trend items are those items needed to maintain longitudinal information relative to basic number and operation knowledge, including student abilities in both computation and estimation. Further, some items might require students to demonstrate estimation skills or mental mathematics without the use of a calculator. Other than these specified items, students should have access to appropriate calculators throughout the test. Many of the new items should be calculator active; that is, they should require the use of a calculator to complete the items.

The recommendation to investigate free use of calculators during the 1996 Mathematics Assessment supports both the philosophy and specific recommendations of NCTM's *Standards*. The availability of such tools can and should provide students with opportunities to demonstrate a higher level of mathematical thinking than they would otherwise be able to exhibit. Further, to deny access to calculators on an assessment is to present an unrealistic picture of the mathematics that students know and are able to do in their real world. There will continue to be an emphasis, in curriculum and in mathematics assessment, on mental mathematics, estimation, number sense, and operation sense, and students will be assessed on knowing when they should use various methods of computation such as mental techniques, pencil and paper, a calculator, or estimation for a given situation. Trend items will continue to measure computational skills in and out of problem contexts. Of primary

importance, students should be expected to apply computational skills in a variety of challenging situations.

Manipulatives

Starting with the 1990 assessment, students were provided rulers and protractors for use in some tasks on the assessments. With the 1992 assessment, students received some geometric shapes to use in responding to items requiring the analysis of relationships between these shapes and more complex shapes that could be formed from the pieces. Assessments in 1996 and beyond should expand this practice, especially in settings where students are given extended time to work with materials that can be easily included in such a large-scale assessment.

Chapter Three

1996 NAEP Mathematics Objectives

Mathematical Content Areas and Assessment Strands

To conduct a meaningful assessment of mathematics proficiency, it is necessary to measure students' proficiencies in various content areas. As in the 1990 and 1992 assessments, five mathematical strands will be used to categorize mathematical content for the 1996 mathematics assessment. The strands are illustrated later in this chapter. Classification of topics into these strands cannot be exact, however, and inevitably involves some overlap. For example, some topics appearing under Data Analysis, Statistics, and Probability may be closely related to others that appear under Algebra and Functions. As assessment programs continue to be refined, it becomes less desirable to force every item into only one content description category. As described in the NCTM *Standards*, students are expected to solve problems that naturally involve more than one specific mathematical topic. Consequently, the assessment as a whole will address the topics and subtopics identified in this chapter, and every item will be categorized under primarily one topic and subtopic so that analysis of results may be somewhat specific. Ideally, however, the items will involve students in synthesizing knowledge across topics and subtopics, and occasionally it may be difficult to identify a unique topic for each item. In fact, it is desirable that at least half of the new items for the 1996 assessment should involve content from more than one topic, or even from more than one strand.

The following sections of this chapter provide a brief description of each content strand with a list of topics and subtopics illustrative of those to be included in the assessment. This level of specificity is needed to guide item writers and ensure adequate coverage of the content areas and abilities to be assessed. The five content strands are largely consistent with the strands used in the 1990 and 1992 assessments. The titles and emphases of the content areas have been modified to reflect more clearly the directions for curriculum and evaluation described in the NCTM *Standards*.

For each of grades 4, 8, and 12, the following symbols are used: a "●" indicates that the subtopic could be assessed at that grade level, a "△" indicates that the subtopic should not be assessed at that grade level, and a "#" indicates that the subtopic might be introduced in the assessment at a very simple level, probably using a manipulative or pictorial model. The test specifications include additional detail and descriptions of how item types, families, calculators, manipulatives, and special studies fit within and across topics and subtopics.

Number Sense, Properties, and Operations

This strand focuses on students' understanding of numbers (whole numbers, fractions, decimals, integers, real numbers, and complex numbers), operations, and estimation, and their application to real-world situations. Students will be expected to demonstrate an understanding of numerical relationships as expressed in ratios, proportions, and percents. Students also will be expected to understand properties of numbers and operations, generalize from numerical patterns, and verify results.

Number sense includes items that address a student's understanding of relative size, equivalent forms of numbers, and his or her use of numbers to represent attributes of real-world objects and quantities. Items that call for students to complete open sentences involving basic number facts are considered part of this content area. Items that require some application of the definition of operations and related procedures are classified under the area of Algebra and Functions.

As in the NCTM *Standards*, the emphasis in computation is on understanding when to use an operation, knowing what the operation means, and being able to estimate and use mental techniques, in addition to performing calculations using computational algorithms. In terms of actual computation, students will be expected to demonstrate that they know how to perform basic algorithms and use calculators in appropriate ways, given more complex situations. While a few isolated computation items may be included, a priority will be placed on including items in which operations are used in meaningful contexts.

The grade 4 assessment will emphasize the development of number sense through the connection of a variety of models to their

numerical representations, as well as emphasizing an understanding of the meaning of addition, subtraction, multiplication, and division. These concepts will be addressed for whole numbers, simple fractions, and decimals at this grade level, with continual emphasis on the use of models and their connection to the use of symbols.

The grade 8 assessment will include number sense extended to include both positive and negative numbers and will address properties and operations involving whole numbers, fractions, decimals, integers, and rational numbers. The use of ratios and proportional thinking to represent situations involving quantity is a major focus at this grade level, and students will be expected to know how to read, use, and apply scientific notation to represent large and small numbers.

At grade 12, the assessment will include both real and complex numbers and will allow students to demonstrate competency through approximately the precalculus or calculus level. Operations with powers and roots, as well as a variety of real and complex numbers, may be assessed. Including a broad range of items at this level will ensure that students who have had different types of high school mathematics courses will be able to demonstrate proficiency on some parts of this content area.

1996 NAEP MATHEMATICS CONTENT STRAND 1

Number Sense, Properties, and Operations	Grade 4	Grade 8	Grade 12
1. Relate counting, grouping, and place value			
a. Use place value to model and describe whole numbers and decimals	●	●	●
b. Use scientific notation in meaningful contexts	△	●	●
2. Represent numbers and operations in a variety of equivalent forms using models, diagrams, and symbols.			
a. Model numbers using set models such as counters	○	△	△

○ Subtopic can be assessed at this grade level.
△ Subtopic should not be assessed at this grade level.
Subtopic may be introduced at a simple level (e.g., using a manipulative or pictorial model).

Number Sense, Properties, and Operations	Grade 4	8	12
b. Model numbers using number lines	●	●	△
c. Use two- and three-dimensional region models to describe numbers	●	●	●
d. Use other models appropriate to a given situation (e.g., draw diagrams to represent a number or an operation; write a number sentence to fit a situation or describe a situation to fit a number sentence; interpret calculator or computer displays)	●	●	●
e. Read, write, rename, order, and compare numbers	●	●	●
3. Compute with numbers (i.e., add, subtract, multiply, divide)			
a. Apply basic properties of operations	●	●	●
b. Describe effect of operations on size and order of numbers	●	●	●
c. Describe features of algorithms (e.g., regrouping with or without manipulatives, partial products)	●	●	●
d. Select appropriate computation method (e.g., pencil and paper, calculator, mental arithmetic)	●	●	●
4. Use computation and estimation in applications			
a. Round whole numbers, decimals, and fractions in meaningful contexts	●	●	●
b. Make estimates appropriate to a given situation			
i. Know when to estimate	●	●	●
ii. Select appropriate type of estimate (overestimate, underestimate, range of estimate)	●	●	●
iii. Describe order of magnitude (estimation related to place value; scientific notation)	●	●	●

● Subtopic can be assessed at this grade level.
△ Subtopic should not be assessed at this grade level.
Subtopic may be introduced at a simple level (e.g., using a manipulative or pictorial model).

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Number Sense, Properties, and Operations	Grade 4	Grade 8	Grade 12
c. Select appropriate method of estimation (e.g., front end, rounding)	●	●	●
d. Solve application problems involving numbers and operations, using exact answers or estimates, as appropriate	●	●	●
e. Interpret round-off errors using calculators/computers (i.e., truncating)	△	#	●
f. Verify solutions and determine the reasonableness of results			
i. in real-world situations	●	●	●
ii. in abstract settings	△	△	●
5. Apply ratios and proportional thinking in a variety of situations			
a. Use ratios to describe situations	#	●	●
b. Use proportions to model problems	△	●	●
c. Use proportional thinking to solve problems (including rates, scaling, and similarity)	△	●	●
d. Understand the meaning of percent (including percents greater than 100 and less than 1)	#	●	●
e. Solve problems involving percent	△	●	●
6. Use elementary number theory			
a. Describe odd and even numbers and their characteristics	●	●	●
b. Describe number patterns	#	●	●
c. Use factors and multiples to model and solve problems	△	●	●
d. Describe prime numbers	△	●	●
e. Use divisibility and remainders in problem settings (including simple modular arithmetic)	△	#	●

● Subtopic can be assessed at this grade level.
△ Subtopic should not be assessed at this grade level.
Subtopic may be introduced at a simple level (e.g., using a manipulative or pictorial model).

Measurement

The measurement strand focuses on an understanding of the process of measurement and on the use of numbers and measures to describe and compare mathematical and real-world objects. Students will be asked to identify attributes, select appropriate units and tools, apply measurement concepts, and communicate measurement-related ideas.

Students should understand and be able to use the measurement attributes of length, mass/weight, capacity, time, money, and temperature. Students will demonstrate their ability to extend basic concepts in applications involving, for example, perimeter, area, surface area, volume, and angle measure.

Students will use measuring instruments and apply measurement concepts to solve problems. Due to the inherent imprecision of measurement tools, it is important for students to recognize that measurement is an approximation.

When students use technology for calculations with imprecise measurements, errors are often carried or increased. Students need to be assessed on their judgments about such answers.

Of these measurement concepts, the focus at grade 4 is on time, money, temperature, length, perimeter, area, capacity, weight/mass, and angle measure. While assessment at grades 8 and 12 continues to include these measurement concepts, the focus shifts to more complex measurement problems that involve volume or surface area or that require students to combine shapes, translate, and apply measures. Students at these grade levels also should solve problems involving proportional thinking (such as scale drawing or map reading) and do applications that involve the use of complex measurement formulas. When appropriate and possible, measurement will be assessed with real measuring devices.

Items requiring straightforward computation with measures, especially those involving time and money, are included not as part of this content area but as a part of Number Sense, Properties, and Operations, instead.

Applications involving measurement provide a rich source for families of questions that illustrate the connections among number sense and operations, algebra, and geometry.

1996 NAEP MATHEMATICS CONTENT STRAND 2

Measurement	Grade 4	Grade 8	Grade 12
1. Estimate the size of an object or compare objects with respect to a given attribute (e.g., length, area, capacity, volume, and weight/mass)	●	●	●
2. Select and use appropriate measurement instruments (e.g., manipulatives such as ruler, meter stick, protractor, thermometer, scales for weight or mass, and gauges)	●	●	●
3. Select and use appropriate units of measurement, according to two criteria:			
a. Type of unit	●	●	●
b. Size of unit	●	●	●
4. Estimate, calculate (using basic principles or formulas), or compare perimeter, area, volume, and surface area in meaningful contexts to solve mathematical and real-world problems			
a. Solve problems involving perimeter and area (e.g., triangles, quadrilaterals, other polygons, circles, and combined forms) [Note: Grade 4 tasks done with manipulatives]	#	●	●
b. Solve problems involving volume and surface area (e.g., rectangular solids, cylinders, cones, pyramids, prisms, and combined forms) [Note: Grades 4 and 8 use manipulatives]	#	#	●

● Subtopic can be assessed at this grade level.
△ Subtopic should not be assessed at this grade level.
Subtopic may be introduced at a simple level (e.g., using a manipulative or pictorial model).

Measurement	Grade 4	8	12
5. Apply given measurement formulas for perimeter, area, volume, and surface area in problem settings	△	●	●
6. Convert from one measurement to another within the same system (customary or metric)	△	●	●
7. Determine precision, accuracy, and error			
a. Apply significant digits in meaningful contexts	△	●	●
b. Determine appropriate size of unit of measurement in problem situation	△	●	●
c. Apply concepts of accuracy of measurement in problem situations	△	●	●
d. Apply absolute and relative error in problem situations	△	△	●
8. Make and read scale drawings	△	●	●
9. Select appropriate methods of measurement (e.g., direct or indirect)	●	●	●
10. Apply the concept of rate to measurement situations	△	●	●

● Subtopic can be assessed at this grade level.
△ Subtopic should not be assessed at this grade level.
Subtopic may be introduced at a simple level (e.g., using a manipulative or pictorial model).

Geometry and Spatial Sense

As described in the NCTM *Standards*, spatial sense must be an integral component of the study and assessment of geometry. Understanding spatial relationships allows students to use the dynamic nature of geometry to connect mathematics to their world.

This content area is designed to extend well beyond low-level identification of geometric shapes into transformations and combinations of those shapes. Informal constructions and demonstrations

(including drawing representations), along with their justifications, take precedence over more traditional types of compass-and-straightedge constructions and proofs. While reasoning is addressed throughout all of the content areas, this strand continues to lend itself to the demonstration of reasoning within both formal and informal settings. The extension of proportional thinking to similar figures and indirect measurement is an important connection here.

In grade 4, students are expected to model properties of shapes under simple combinations and transformations, and they are expected to use mathematical communication skills to draw figures given a verbal description. For grade 8, students are expected to have extended their understanding to include properties of angles and polygons and to apply reasoning skills to make and validate conjectures about transformations and combinations of shapes. At grade 12, students are expected to demonstrate proficiency with transformational geometry and to apply concepts of proportional thinking to a variety of geometric situations. They will have opportunities to demonstrate more sophisticated reasoning processes than at earlier grade levels, and they will be expected to demonstrate a variety of algebraic and geometric connections. The importance of these connections and their use in solving problems is indicated by the shifting emphasis in geometry toward coordinate geometry, as described in Chapter Four.

1996 NAEP MATHEMATICS CONTENT STRAND 3

	Grade		
Geometry and Spatial Sense	**4**	**8**	**12**
1. Describe, visualize, draw, and construct geometric figures			
a. Draw or sketch a figure given a verbal description [open-ended items]	●	●	●
b. Given a figure, write a verbal description of its geometric qualities	△	●	●

● Subtopic can be assessed at this grade level.
△ Subtopic should not be assessed at this grade level.
Subtopic may be introduced at a simple level (e.g., using a manipulative or pictorial model).

Geometry and Spatial Sense	Grade 4	Grade 8	Grade 12
2. Investigate and predict results of combining, subdividing, and changing shapes (e.g., paper folding, dissecting, tiling, and rearranging pieces of solids)	●	●	●
3. Identify the relationship (congruence, similarity) between a figure and its image under a transformation			
a. Use motion geometry (informal: lines of symmetry, flips, turns, and slides)	●	●	●
b. Use transformations (translations, rotations, reflections, dilations, and symmetry)			
i. Synthetic	△	#	●
ii. Algebraic	△	△	●
4. Describe the intersection of two or more geometric figures			
a. Two dimensional	△	●	●
b. Planar cross-section of a solid	△	●	●
5. Classify figures in terms of congruence and similarity, and informally apply these relationships using proportional reasoning where appropriate	△	●	●
6. Apply geometric properties and relationships in solving problems			
a. Use concepts of 'between,' 'inside,' 'on,' and 'outside'	●	●	△
b. Use the Pythagorean relationship to solve problems	△	●	●
c. Apply properties of ratio and proportion with respect to similarity	△	#	●
e. Solve problems involving right triangle trigonometric applications	△	△	●

● Subtopic can be assessed at this grade level.
△ Subtopic should not be assessed at this grade level.
Subtopic may be introduced at a simple level (e.g., using a manipulative or pictorial model).

Geometry and Spatial Sense	Grade 4	8	12
7. Establish and explain relationships involving geometric concepts			
a. Make conjectures	●	●	●
b. Validate and justify conclusions and generalizations	●	●	●
c. Use informal induction and deduction	#	●	●
8. Represent problem situations with geometric models and apply properties of figures in meaningful contexts to solve mathematical and real-world problems	●	●	●
9. Represent geometric figures and properties algebraically using coordinates and vectors			
a. Use properties of lines (including distance, midpoint, slope, parallelism and perpendicularity) to describe figures algebraically	△	#	●
b. Algebraically describe conic sections and their properties	△	△	●
c. Use vectors in problem situations (addition, subtraction, scalar multiplication, dot product)	△	△	●

● Subtopic can be assessed at this grade level.

△ Subtopic should not be assessed at this grade level.

Subtopic may be introduced at a simple level (e.g., using a manipulative or pictorial model).

Data Analysis, Statistics, and Probability

Because of its fundamental role in making sense of the world, this content area will receive increased emphasis. The important skills of collecting, organizing, reading, representing, and interpreting data will be assessed in a variety of contexts to reflect the pervasive use of these skills in dealing with information. Statistics and statistical concepts extend these basic skills to include analyzing and communicating increasingly sophisticated interpretations of data. Dealing with uncertainty and making predictions about

outcomes require an understanding not only of the meaning of basic probability concepts but also the application of those concepts in problem-solving and decision-making situations.

Questions will emphasize appropriate methods for gathering data, the visual exploration of data, a variety of ways of representing data, and the development and evaluation of arguments based on data analysis. Students will be expected to apply these ideas in increasingly sophisticated situations that require increasingly comprehensive analysis and decision making.

For grade 4, students will be expected to apply their understanding of number and quantity by solving problems involving data, and they will use data analysis to broaden their number sense. They will be expected to be familiar with a variety of types of graphs. They will be asked to make predictions from data and explain their reasoning, and they will deal informally with measures of central tendency. Grade 4 students will also use the basic concept of chance in meaningful contexts not involving the computation of probabilities.

Probabilistic thinking and a variety of specialized graphs become increasingly important in grades 8 and 12. Students in grade 8 will be expected to analyze statistical claims and design experiments, and they may use simulations to model real-world situations. They should have some understanding of sampling, and they should be asked to make predictions based on experiments or data. They will begin to use some formal terminology related to probability, data analysis, and statistics. By grade 8, students should be comfortable with a variety of types of graphs to represent different types of data in different situations.

Students in grade 12 will be expected to use a wide variety of statistical techniques to model situations and solve problems. Students at this level should apply concepts of probability to explore dependent and independent events, and they should be somewhat knowledgeable about conditional probability. They should be able to use formulas and more formal terminology to describe a variety of situations. By this level, students should have a basic understanding of the use of mathematical equations and graphs to interpret data, including the use of curve fitting to match a set of data with an appropriate mathematical model.

1996 NAEP MATHEMATICS CONTENT STRAND 4

Data Analysis, Statistics, and Probability	Grade 4	Grade 8	Grade 12
1. Read, interpret, and make predictions using tables and graphs			
a. Read and interpret data	●	●	●
b. Solve problems by estimating and computing with data	●	●	●
c. Interpolate or extrapolate from data	△	●	●
2. Organize and display data and make inferences			
a. Use tables, histograms (bar graphs), pictograms, and line graphs	●	●	●
b. Use circle graphs and scattergrams	△	●	●
c. Use stem-and-leaf plots and box-and-whisker plots	△	●	●
d. Make decisions about outliers	△	●	●
3. Understand and apply sampling, randomness, and bias in data collection			
a. Given a situation, identify sources of sampling error	△	●	●
b. Describe a procedure for selecting an unbiased sample	△	●	●
c. Make generalizations based on sample results	△	●	●
4. Describe measures of central tendency and dispersion in real-world situations	#	●	●
5. Use measures of central tendency, correlation, dispersion, and shapes of distributions to describe statistical relationships			
a. Use standard deviation and variance	△	△	●
b. Use the standard normal distribution	△	△	●
c. Make predictions and decisions involving correlation	△	△	●

● Subtopic can be assessed at this grade level.
△ Subtopic should not be assessed at this grade level.
Subtopic may be introduced at a simple level (e.g., using a manipulative or pictorial model).

Data Analysis, Statistics, and Probability	Grade 4	Grade 8	Grade 12
6. Understand and reason about the use and misuse of statistics in our society			
a. Given certain situations and reported results, identify faulty arguments or misleading presentations of the data	#	●	●
b. Appropriately apply statistics to real-world situations	#	●	●
7. Fit a line or curve to a set of data and use this line or curve to make predictions about the data, using frequency distributions where appropriate	△	△	●
8. Design a statistical experiment to study a problem and communicate the outcomes	△	●	●
9. Use basic concepts, trees, and formulas for combinations, permutations, and other counting techniques to determine the number of ways an event can occur	△	●	●
10. Determine the probability of a simple event			
a. Estimate probabilities by use of simulations	△	●	●
b. Use sample spaces and the definition of probability to describe events	●	●	●
c. Describe and make predictions about expected outcomes	△	●	●
11. Apply the basic concept of probability to real-world situations			
a. Informal use of probabilistic thinking	●	●	●
b. Use probability related to independent and dependent events	△	●	●
c. Use probability related to simple and compound events	△	△	●
d. Use conditional probability	△	△	●

● Subtopic can be assessed at this grade level.
△ Subtopic should not be assessed at this grade level.
Subtopic may be introduced at a simple level (e.g., using a manipulative or pictorial model).

Algebra and Functions

This strand extends from work with simple patterns at grade 4, to basic algebra concepts at grade 8, to sophisticated analysis at grade 12, and involves not only algebra but also precalculus and some topics from discrete mathematics. As described in the NCTM *Standards*, these algebraic concepts are developed throughout the grades with informal modeling done at the elementary level and with increased emphasis on functions at the secondary level. The nature of the algebraic concepts and procedures included in the assessment at all levels will reflect the NCTM *Standards*. Students will be expected to use algebraic notation and thinking in meaningful contexts to solve mathematical and real-world problems, specifically addressing an increasing understanding of the use of functions (including algebraic and geometric) as a representational tool.

The assessment at all levels will include the use of open sentences and equations as representational tools. Students will use the notion of equivalent representations to transform and solve number sentences and equations of increasing levels of complexity.

The grade 4 assessment will involve informal demonstration of students' abilities to generalize from patterns, including the justification of their generalizations. Students will be expected to translate between mathematical representations, to use simple equations, and to do basic graphing.

At grade 8, the assessment will include more algebraic notation, stressing the meaning of variable and an informal understanding of the use of symbolic representations in problem-solving contexts. Students at this level will be asked to use variables to represent a rule underlying a pattern. They should have a beginning understanding of equations as a modeling tool, and they should solve simple equations and inequalities by a variety of methods, including both graphical and basic algebraic methods. Students should begin to use basic concepts of functions as a way of describing relationships.

By grade 12, students will be expected to be adept at appropriately choosing and applying a rich set of representational tools in a variety of problem-solving situations. They should have an understanding of basic algebraic notation and terminology as they relate

to representations of mathematical and real-world problem situations. Students should be able to use functions as a way of representing and describing relationships.

1996 NAEP MATHEMATICS CONTENT STRAND 5

Algebra and Functions	Grade 4	Grade 8	Grade 12
1. Describe, extend, interpolate, transform, and create a wide variety of patterns and functional relationships			
a. Recognize patterns and sequences	●	●	●
b. Extend a pattern or functional relationship	●	●	●
c. Given a verbal description, extend or interpolate with a pattern (complete a missing term)	△	●	●
d. Translate patterns from one context to another	#	●	●
e. Create an example of a pattern or functional relationship	●	●	●
f. Understand and apply the concept of a variable	#	●	●
2. Use multiple representations for situations to translate among diagrams, models, and symbolic expressions	●	●	●
3. Use number lines and rectangular coordinate systems as representational tools			
a. Identify or graph sets of points on a number line or in a rectangular coordinate system	●	●	●
b. Identify or graph sets of points in a polar coordinate system	△	●	●
c. Work with applications using coordinates	△	●	●
d. Transform the graph of a function	△	#	●

● Subtopic can be assessed at this grade level.
△ Subtopic should not be assessed at this grade level.
Subtopic may be introduced at a simple level (e.g., using a manipulative or pictorial model).

Algebra and Functions	Grade 4	Grade 8	Grade 12
4. Represent and describe solutions to linear equations and inequalities to solve mathematical and real-world problems			
a. Solution sets of whole numbers	●	●	●
b. Solution sets of real numbers	#	●	●
5. Interpret contextual situations and perform algebraic operations on real numbers and algebraic expressions to solve mathematical and real-world problems			
a. Perform basic operations, using appropriate tools, on real numbers in meaningful contexts (including grouping and order of multiple operations involving basic operations, exponents, and roots)	△	●	●
b. Solve problems involving substitution in expressions and formulas	△	●	●
c. Solve meaningful problems involving a formula with one variable	△	●	●
d. Use equivalent forms to solve problems	△	●	●
6. Solve systems of equations and inequalities using appropriate methods			
a. Solve systems graphically	△	●	●
b. Solve systems algebraically	△	△	●
c. Solve systems using matrices	△	△	●
7. Use mathematical reasoning			
a. Make conjectures	●	●	●
b. Validate and justify conclusions and generalizations	●	●	●
c. Use informal induction and deduction	#	●	●
8. Represent problem situations with discrete structures			
a. Use finite graphs and matrices	△	#	●
b. Use sequences and series	△	△	●

● Subtopic can be assessed at this grade level.
△ Subtopic should not be assessed at this grade level.
Subtopic may be introduced at a simple level (e.g., using a manipulative or pictorial model).

Algebra and Functions	Grade 4	Grade 8	Grade 12
c. Use recursive relations (including numerical and graphical iteration and finite differences)	△	△	●
9. Solve polynomial equations with real and complex roots using a variety of algebraic and graphical methods and using appropriate tools	△	△	●
10. Approximate solutions of equations (bisection, sign changes, and successive approximations)	△	#	●
11. Use appropriate notation and terminology to describe functions and their properties (including domain, range, function composition, and inverses)	△	△	●
12. Compare and apply the numerical, symbolic, and graphical properties of a variety of functions and families of functions, examining general parameters and their effect on curve shape	△	#	●
13. Apply function concepts to model and deal with real-world situations	△	#	●
14. Use trigonometry			
a. Use triangle trigonometry to model problem situations	△	△	●
b. Use trigonometric and circular functions to model real-world phenomena	△	△	●
c. Apply concepts of trigonometry to solve real-world problems	△	△	●

● Subtopic can be assessed at this grade level.
△ Subtopic should not be assessed at this grade level.
Subtopic may be introduced at a simple level (e.g., using a manipulative or pictorial model).

Chapter Four

Cognitive Abilities

While NAEP was designed to monitor, assess, and report student achievement nationally, an inevitable effect of this monitoring and reporting is clearly improvement in mathematics learning. The NCTM *Curriculum and Evaluation Standards for School Mathematics* acknowledges that if real change in the mathematics curriculum is to take place, the manner in which assessment is conducted will also have to change. In classrooms, assessment activities often are the primary sources from which students discern what teachers really value and what teachers really want them to know. As a result, over time, the portions of the curriculum that are tested become the portions of the curriculum that receive greater emphasis both in teachers' and in students' allocations of effort and time.

Mathematical Power

Central to the NCTM *Standards* description of the features of students' performance that should be assessed is "mathematical power." Mathematical power is characterized as a student's overall ability to gather and use mathematical knowledge through exploring, conjecturing, and reasoning logically; through solving nonroutine problems; through communicating about and through mathematics; and through connecting mathematical ideas in one context with mathematical ideas in another context or with ideas from another discipline in the same or related contexts.

Assessing a student's mathematical power requires many different indicators over time. As power develops beyond the general mathematical abilities of conceptual understanding, procedural knowledge, and problem solving, it is important to ensure that measures are taken of a student's ability to *reason* in mathematical situations, to *communicate* perceptions and conclusions drawn from a mathematical context, and to *connect* the mathematical nature of a situation with related mathematical knowledge and information gained from other disciplines or through observation.

It is the total interaction of all of these abilities that defines a student's overall mathematical power at a given time. The mental skills of reasoning, communicating, and connecting lie at the foundation of each of the content strands and each of the mathematical abilities featured in prior NAEP assessments. These relationships, illustrated in Chapter Two, indicate the multidimensional nature of mathematical power.

Mathematical power can be viewed from a variety of perspectives. Students may encounter a new problem in an old context or an old problem in a new context. When first attempts to solve a problem fail, the student may reexamine the information, rework it, and then reapply it to the situation in a more productive fashion. The process of revising an approach to a problem based on reasoning, gathering new information, and making connections with other ideas is a dynamically growing and changing ability. This feature of mathematical power can be viewed through student performance within a particular content strand at the conceptual, procedural, and problem-solving levels of ability. Equally, a particular concept, procedure, or problem context might be viewed across the different strands. In the latter case, families of items are particularly helpful in assessment. The use of hand calculators allows students to quickly pursue alternative paths and check to see if they either provide fruitful new information or reconfirm judgments made through other approaches.

Students display their mathematical power through the formulation of lines of attack on problems and the way in which they reason through situations involving a multitude of possibilities. It is here that the NCTM recommendation that students experience a number of extended open-ended items requiring construction of responses is important. Through a student's report of his or her thinking, the questions of the relevance of approach, nature of reasoning, and ability to solve problems becomes less a high inference guess and more a conclusion that can be drawn from evidence. This is especially true when the collected evidence includes the communication of a student's approach and when partial credit for student efforts is awarded in the scoring of an item.

Finally, mathematical power is a function of students' prior knowledge and experience and the ability to connect that knowledge in productive ways to new contexts. This aspect of power can

be measured with the multiple-choice items and through analysis of the ways in which students develop their responses to the constructed response items on the assessment.

Information related to these features of students' development is as difficult to isolate and statistically extract from the data as the mathematical abilities featured in the past NAEP assessments in mathematics. However, they are important aspects of the mathematical development of students. As such, the three features of mathematical power (reasoning, communication, and connections) will be used as underlying threads for item construction and overall test design. In 1996, these threads may not be specifically reported, although they will be represented in the overall way the assessment is conceived and developed.

Mathematical Abilities

The dimensions of general mental abilities associated with mathematics and used in past NAEP assessments are conceptual understanding, procedural knowledge, and problem solving. These three areas are specifically identified by the *Standards* as primary foci for assessment, and they received focal attention in the design of the 1990 and 1992 assessments. Conceptual understanding can be viewed simply as a measure of a student's knowing "that" or "about," while procedural knowledge can be viewed as a student's knowing "how." These two abilities combined provide a base for the capability to recognize and understand a situation, to formulate a plan to confront the situation, to arrive at a solution to the problem the situation presents, and to reflect upon the solution. These latter stages can be thought of as facets of problem solving.

However, as recommended in Chapter One, the role these dimensions of students' mathematical power will play in the 1996 assessment should change from one of a direct matrix feature to one of a design characteristic that assists in providing balance to the overall assessment. The NAEP design for the 1996 assessment should certainly continue to focus on conceptual understanding, procedural knowledge, and problem solving in bringing some balance to the assessments for grades 4, 8, and 12. In particular, it is recommended that the overall mixture of assessment items for each grade level include at least one-third of the items measuring each of the abilities of conceptual understanding, procedural knowledge, and problem solving.

As with the mathematical content strands, mathematical abilities are not separate and distinct factors of an individual's ways of thinking about a mathematical situation. These abilities are, rather, descriptions of the ways in which information is structured for instruction and the ways in which students manipulate, reason with, or communicate their mathematical ideas. As a consequence, there can be no singular or unanimous agreement among educators about what constitutes a conceptual, a procedural, or a problem-solving item. What can be classified are the actions a student is likely to undertake in processing information and providing a satisfactory response. Thus, within the content strands, assessment tasks will be classified according to the ability categories they most closely represent in terms of the type of processing they might be expected to require. Further, the mathematical power features of reasoning, communication, and connections will be woven through the specifications to provide an added level of richness to the 1996 assessment tasks.

The following discussions of conceptual understanding, procedural knowledge, and problem solving are given to illustrate the primary features the NAEP assessment should employ in trying to capture features of cognitive activities that combine to empower a student in mathematical situations.

Conceptual Understanding

Students demonstrate conceptual understanding in mathematics when they provide evidence that they can recognize, label, and generate examples and nonexamples of concepts; use and interrelate models, diagrams, manipulatives, and varied representations of concepts; identify and apply principles (i.e., valid statements generalizing relationships among concepts in conditional form); know and apply facts and definitions; compare, contrast, and integrate related concepts and principles to extend the nature of concepts and principles; recognize, interpret, and apply the signs, symbols, and terms used to represent concepts; or interpret the assumptions and relations involving concepts in mathematical settings.

Conceptual understanding reflects a student's ability to reason in settings involving the careful application of concept definitions, relations, or representations of either. Such an ability is reflected by student performance that indicates the production of examples,

common or unique representations, or communication indicating the ability to manipulate central ideas about the understanding of a concept in a variety of ways.

Procedural Knowledge

Students demonstrate procedural knowledge in mathematics when they select and apply appropriate procedures correctly; verify or justify the correctness of a procedure using concrete models or symbolic methods; or extend or modify procedures to deal with factors inherent in problem settings.

Procedural knowledge includes the various numerical algorithms in mathematics that have been created as tools to meet specific needs efficiently. Procedural knowledge also encompasses the abilities to read and produce graphs and tables, execute geometric constructions, and perform noncomputational skills such as rounding and ordering. These latter activities can be differentiated from conceptual understanding by the task context or presumed student background—that is, an assumption that the student has the conceptual understanding of a representation and can apply it as a tool to create a product or to achieve a numerical result. In these settings, the assessment question is how well the student executed a procedure or how well the student selected the appropriate procedure to effect a given task.

Procedural knowledge is often reflected in a student's ability to connect an algorithmic process with a given problem situation, to employ that algorithm correctly, and to communicate the results of the algorithm in the context of the problem setting. Procedural understanding also encompasses a student's ability to reason through a situation, describing why a particular procedure will give the correct answer for a problem in the context described.

Problem Solving

In problem solving, students are required to use their accumulated knowledge of mathematics in new situations. Problem solving requires students to recognize and formulate problems; determine the sufficiency and consistency of data; use strategies, data, models, and relevant mathematics; generate, extend, and modify procedures; use reasoning (i.e., spatial, inductive, deductive, statistical, or proportional) in new settings; and judge the reasonableness and correctness of solutions. Problem solving situations require students

to connect all of their mathematical knowledge of concepts, procedures, reasoning, and communication/representational skills in confronting new situations. As such, these situations are, perhaps, the most accurate measures of students' proficiency in mathematics.

Chapter Five

Item Types

Multiple-Choice Items

Central to the development of the NAEP assessment in mathematics is the careful selection of items/tasks. At present, they consist primarily of either multiple-choice items or short answer, open-ended items. Multiple-choice items require students to read, reflect, or compute and then to select the alternative that best expresses what they believe the answer to be. An example from the 1990 NAEP assessment follows:

Which of the following is true about 87 percent of 10?

A. It is greater than 10.
B. It is less than 10.
C. It is equal to 10.
D. Can't tell.
E. I don't know.

The actual sequence of conceptual, procedural, reasoning, and problem-solving skills that a student might employ in responding to this question is impossible to determine. However, the best judgment of experts is that an 8th-grade student would probably consider the item at a procedural level, reflecting on the meaning of percent, multiplying 10 by .87, and then, after comparing the response of 8.7 with each of the alternatives, selecting choice B. A 12th grader might rely on his or her knowledge that 87 percent of a quantity is smaller than the quantity itself and therefore B is the correct answer. The work at the 8th-grade level might be more procedural in nature; the work at the 12th grade more conceptual. Both of these approaches resulted in a correct answer. Hence, it is dangerous to assume that the way an item is cast or classified will always reflect the way in which a student actually addresses it.

Open-Ended Items

To provide more reliable and valid opportunities for extrapolating about students' approaches to problems, recent NAEP assessments have included items that are often referred to as open ended. These are short-answer items that require students to give either a numerical result or the correct name or classification for a group of mathematical objects, draw an example of a given concept, or, perhaps, write a brief explanation for a given result. An example, taken from the 4th-grade 1992 NAEP assessment follows:

> Lynn had only quarters, dimes, and nickels to buy her lunch. She spent all of the money and received no change. Could she have spent $1.98?
>
> Yes No
>
> Give a reason for your answer.

This item requires that children reflect on the situation, provide their own computation or answer, and then communicate the reasoning behind their solution. In scoring, open-ended items allow for the awarding of credit for more than one response. They also allow for crediting growth in student knowledge (e.g., awarding partial credit). They also enrich the analysis of the assessment by generating data that are more detailed in descriptions of student abilities and mathematical misconceptions.

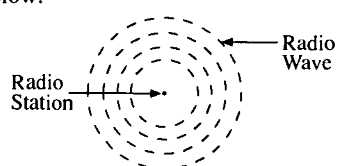
However more powerful than multiple-choice items the open-ended items may be, they still do not provide information about the level of change in children's reasoning, problem-solving, or communication skills. For this reason, starting in 1992, the NAEP mathematics assessment included extended open-ended tasks.

Extended Open-Ended Items

Extended open-ended items require students to consider a situation that demands more than a numerical response or a short verbal communication. These items require the student to carefully consider a situation within, or across, the content strand areas, understand what is required to "solve" the situation, choose a plan

of attack, carry out the attack, and interpret the solution derived in terms of the original situation. The response mode requires that students provide evidence of their work on some of these aspects of the solving process and communicate their decision-making steps in the context of the problem.

For example, consider the following problem developed for the 1992 NAEP mathematics assessment:

> This question requires you to show your work and explain your reasoning. You may use drawings, words, and numbers in your explanation. Your answer should be clear enough so that another person could read it and understand your thinking. It is important that you show <u>all</u> your work.
>
> Radio station KMAT in Math City is 200 miles from radio station KGEO in Geometry City. Highway 7, a straight road, connects the two cities.
>
> KMAT broadcasts can be received up to 150 miles in all directions from the station and KGEO broadcasts can be received up to 125 miles in all directions. Radio waves travel from each radio station through the air, as represented below.
>
> 
>
>
> On the next page, draw a diagram that shows the following.
>
> - Highway 7
> - The location of the two radio stations
> - The part of Highway 7 where both radio stations can be received
>
> Be sure to label the distances along the highway and the length in miles of the part of the highway where both stations can be received.

In this example, students must use logic and diagrams to communicate the reasoning behind their solution to this task. Hence, several key elements of mathematical power are being measured here.

Scoring Extended Open-Ended Items

Extended open-ended items in mathematics should be evaluated according to an established grading scale developed from a sample of actual student responses. The scale used should follow a multiple-point format similar to the following that was developed for scoring the NAEP "radio stations" task described earlier.

Rating	Performance Category
0	No response
1	Incorrect—The work is completely incorrect or irrelevant, or the response states, "I don't know."
2	Minimal—Diagram with only cities, Highway 7, and 200 miles labeled; or a diagram that shows some, but not all of the given distances: 125, 150, or 200 miles. Minimal responses do not recognize that the common broadcast area is a *length* along the highway.
3	Partial—Diagram with cities, Highway 7, and 200 miles labeled *and* identification of common broadcast area as a length along (or not on) the highway. Two or more of the radio wave distances 250, 125, and 75 are insufficiently labeled.
4	Satisfactory—Diagram with cities, Highway 7, 200 miles, and all radio wave distances labeled *and* identification of common broadcast area on Highway 7 as a length. At the same time, omits or incorrectly computes length of the highway along which both radio stations can be received.
5	Extended—An accurate, well-labeled diagram (as described in the score 4 category) clearly indicating that the portion of Highway 7 along which both radio stations can be received is 75 miles in length.

A scoring scale, adapted to a particular problem and applied by experienced scorers, will provide rich information about students' conceptual and procedural understanding of a problem situation and their ability to problem solve and communicate understanding of the process in the context of a problem.

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